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Operator Structure of a Nonquantum and Nonclassical System

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There exists a connection between the vectors of the Poincaré-sphere and the elements of the complex Hilbert space $\mathbb{C}^2$. This latter space is used to describe spin-1/2 measurements. We use this connection to study the intermediate cases of a more general spin-1/2 measurement model which has no representation in a Hilbert space. We construct the set of operators of this general model and investigate under which circumstances it is possible to define linear operators. Because no Hilbert space structure is possible for these intermediate cases, it can be expected that no linear operators are possible and it is shown that under very plausible assumptions this is indeed the case.

1. INTRODUCTION

In previous papers (D. Aerts, 1983, 1986, 1987) one of the authors proposed an explanation for the probabilities of quantum mechanics based on the assumption that quantum structures arise as a consequence of the presence of fluctuations on the interaction between the measurement apparatus and the entity under study. It is shown in this approach, which has been called the "hidden measurement" approach, that quantum probabilities can be reproduced by considering an experiment as a class of subexperiments (the hidden measurements), indistinguishable to the macroscopic observer, and which can be parametrized by a real parameter. The resulting probabilities through the averaging process over the whole class of hidden measurements coincided with the quantum probabilities. A model for the spin-1/2 experiments was introduced, and later generalized to include cases of arbitrary fluctuations, going from maximal fluctuations which coincide with quantum mechanics, to cases of zero fluctuations, which were shown to be classical

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experiments. The amount of fluctuation can be parametrized by a real parameter $\epsilon \in [0, 1]$, and the name ϵ-model was given to the model (D. Aerts *et al.*, 1993a, b). To make this article self-contained we will briefly reprise this ϵ-model in Section 2. By considering $\epsilon \neq 1, 0$ intermediate cases can be found, which are neither quantum ($\epsilon = 1$) nor classical ($\epsilon = 0$). So it was necessary to present a theory (D. Aerts, 1994; D. Aerts and Durt, 1994a, b) which is much more general than quantum mechanics and where these intermediate cases can be examined.

The intermediate cases have already been investigated in several different mathematical categories. The structure of the Piron lattice of properties (Piron, 1976) of the entity was investigated and was found to be Boolean for the classical case with zero fluctuations and pure quantum for the case of maximal fluctuations. For the intermediate cases the lattice was neither Boolean nor quantum (D. Aerts and Durt, 1994a, b). In the category of the closures it was shown (D. Aerts and Durt, 1994a, b) how the superposition principle disappears during the transition from quantum to classical, and that for the intermediate cases the Piron axioms to find a representation of the entity in a general Hilbert space are violated. In another category, that of the probability structures, a transition from Kolmogorovian to non-Kolmogorovian was discovered such that again the intermediate cases are neither Kolmogorovian nor quantum-like (D. Aerts, 1995; S. Aerts, 1996).

In this paper we will investigate the ϵ-model in yet another category, that of observables. More precisely, we will look at the properties of the *-algebra of linear operators which are used in quantum mechanics to describe observables, as we go from the quantum to the classical case by varying the parameter ϵ. In analogy with the disappearing of the superposition principle, we expect to find a loss of the linearity of the operators as we pass from the quantum to the classical case, and rigorous proofs of this presumption will be sought.

2. THE ϵ-MODEL

The physical entity S that we consider is a point particle P that can move on the surface of a sphere, denoted by *surf*, with center O and radius 1. The unit vector v where the particle is located on *surf* represents the state p_v of the particle. Hence the set of states that we consider is given by $\Sigma = \{p_v \mid v \in surf\}$. To introduce the experiments, we consider two diametrically opposite points u and $-u$ on the surface of the sphere, and denote by $[-1, +1]_u$ the interval of real numbers $[-1, +1]$ coordinating the points of the line between u and $-u$ in such a way that -1 coordinates $-u$ and $+1$ coordinates u. We introduce a real parameter $\epsilon \in [0, 1]$ and consider the subinterval $[-\epsilon, +\epsilon]_u \subset [-1, +1]_u$. The experiment e_u^ϵ consists of the particle

P falling from its original place v orthogonally onto the line between u and $-u$ and arriving at a point coordinated in the interval $[-1, +1]_u$ by the real number $v \cdot u$. In the interval $[-\epsilon, +\epsilon]_u$ we consider a uniformly distributed random variable κ, and the experiment proceeds as follows. If $\kappa \in [-\epsilon, v \cdot u[$, the particle P moves to the point u and the experiment e_u^ϵ gives outcome x_u^1. If $\kappa \in]v \cdot u, +\epsilon]$, it moves to the point $-u$ and the experiment e_u^ϵ gives outcome x_u^2. If $\kappa = v \cdot u$, it moves with probability 1/2 to the point u, and the experiment e_u^ϵ gives outcomes x_u^1, and it moves with probability 1/2 to the point $-u$, and the experiment e_u^ϵ gives outcome x_u^2. This completes the description of the experiment e_u^ϵ. We shall denote the set of possible outcomes $\{x_u^1, x_u^2\}$ of the experiment e_u^ϵ by $O_{e_u^\epsilon}$ and the probability to obtain an outcome x_u^1 by performing an experiment e_u^ϵ (respectively an outcome x_u^2) if the entity is in a state p_v by $P(e_u^\epsilon = x_u^1 | p_v)$ [respectively $P(e_u^\epsilon = x_u^2 | p_v)$].

We shall now consider different situations labeled by the parameter ϵ. The entity $S(\epsilon)$ is described by a set of states $\Sigma = \{p_v | v \in surf\}$, a set of experiments $E(\epsilon) = \{e_u^\epsilon | u \in surf, \epsilon \in [0, 1]\}$, and a set of probabilities $\{P(e_u^\epsilon = x_u^i | p_v) | e_u^\epsilon \in E(\epsilon), x_u^i \in O_{e_u^\epsilon}, p_v \in \Sigma\}$. To lighten the notation we denote the probability $P(e_u^\epsilon = x_u^1 | p_v)$ by $P^\epsilon(p_u | p_v)$ and the probability $P(e_u^\epsilon = x_u^2 | p_v)$ by $P^\epsilon(p_{-u} | p_v)$. We have the following cases:

1. $\epsilon \leq v \cdot u$. Then $P^\epsilon(p_u | p_v) = 1$ and $P^\epsilon(p_{-u} | p_v) = 0$.
2. $-\epsilon \leq v \cdot u \leq +\epsilon$. Then

$$P^\epsilon(p_u | p_v) = \frac{1}{2\epsilon}(v \cdot u + \epsilon)$$

$$P^\epsilon(p_{-u} | p_v) = \frac{1}{2\epsilon}(-v \cdot u + \epsilon)$$

3. $v \cdot u \leq -\epsilon$. Then $P^\epsilon(p_u | p_v) = 0$ and $P^\epsilon(p_{-u} | p_v) = 1$.

In the case $\epsilon = 1$, we see that the probabilities coincide with the probability of a spin-1/2 entity in quantum mechanics, and that the entity S can be described in a Hilbert space and the experiments e_u^ϵ by the linear self-adjunct operators of that Hilbert space.

When we vary ϵ over [0, 1], we get intermediate cases going from quantum ($\epsilon = 1$) to classical ($\epsilon = 0$). It is possible to study whether the axioms to derive a Hilbert space structure from the lattice of properties are satisfied for the intermediate cases. It was proven that this is only the case for $\epsilon = 1$. We will now study the ϵ-model in the category of the operators, more precisely in the *-algebra used to describe the operators of Hilbert space quantum mechanics. Because there is no Hilbert space structure available for the intermediate cases, we need other guidelines. The most natural guidelines we can think of are based on physical observables and mathematical simplicity.

3. THE MAP FROM POINCARÉ SPHERE TO A SET OF EIGENVECTORS

We can establish a correspondence between the points of the unit-sphere in $\mathbb{R}^3$, which we will call the Poincaré sphere, and the eigenvectors of the spin-1/2 operators in the complex space $\mathbb{C}^4$ in the following way. The mapping

$$S\colon\ \mathbb{R}^3 \to \mathbb{C}^4\colon\quad u = (\sin\theta\cos\varphi,\quad \sin\theta\sin\varphi,\quad \cos\theta)$$
$$\to S_u = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$$

maps a unit vector u on the spin-1/2 operator S_u which has two eigenvectors, namely

$$s_{u+} = \begin{pmatrix} \cos\dfrac{\theta}{2}\, e^{-i\varphi/2} \\ \\ \sin\dfrac{\theta}{2}\, e^{i\varphi/2} \end{pmatrix} \qquad \text{and} \qquad s_{u-} = \begin{pmatrix} -\sin\dfrac{\theta}{2}\, e^{-i\varphi/2} \\ \\ \cos\dfrac{\theta}{2}\, e^{i\varphi/2} \end{pmatrix}$$

with eigenvalue $+1$ and -1, respectively. The set of these two orthogonal eigenvectors is a basis for the complex space $\mathbb{C}^2$. We can attribute the following meaning to these eigenvectors: if the entity is in a state s_{u+} we will find the value $+1$ with certainty. The interpretation is then that on the Poincaré sphere the entity is in the state p_u given by the point u. In short, we make a unit vector u in $\mathbb{R}^3$ correspond with an eigenvector s_{u+} in $\mathbb{C}^2$. This correspondence can be made one-to-one, as we shall show now.

Because we are switching from the space of reals $\mathbb{R}^3$ to the complex space $\mathbb{C}^2$ we will not use the word "dimensions," but the expression "degrees of freedom." On the Poincaré sphere we have two degrees of freedom: θ and φ. In the complex space $\mathbb{C}^2$ there are four: each complex number can be written as the sum of its real part and its imaginary part; but by demanding that the norm of the eigenvector is 1, and because an eigenvector is defined upon an arbitrary constant (following the first requirement of modulus 1), we have a one-to-one correspondence between the vector of unit length u and the set of eigenvectors of S_u with eigenvalue 1 and norm 1. It is obvious that there exists a map between the Poincaré sphere and the set of eigenvectors of S_u with eigenvalue 1 and norm 1 by the foregoing remarks. The inverse map also exists. Indeed, if we take an arbitrary element of $\mathbb{C}^2$ with norm 1, we can always construct one and only one unit vector corresponding to this element, invariant under multiplication of this element by an arbitrary complex number of modulus 1. Let us take

$$s = \begin{pmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{pmatrix} \qquad \text{with} \quad r_1^2 + r_2^2 = 1 \text{ and } r_1, r_2 \in \mathbb{R}^{+}$$

Then we can let this coincide with

$$s_{u+} = \begin{pmatrix} \cos\dfrac{\theta}{2}\, e^{-i\varphi/2} \\ \\ \sin\dfrac{\theta}{2}\, e^{i\varphi/2} \end{pmatrix}$$

by

$$\begin{cases} e^{i\theta_1} = e^{-i\varphi/2} \\ e^{i\theta_2} = e^{i\varphi/2} \end{cases} \Rightarrow \theta_2 - \theta_1 = \varphi + k\cdot 2\pi; \quad k \in \mathbb{Z}$$

$$\begin{cases} r_1 = \cos\dfrac{\theta}{2} \\ r_2 = \sin\dfrac{\theta}{2} \end{cases} \Rightarrow \tan\frac{\theta}{2} = \frac{r_2}{r_1}$$

We see that multiplication of s with an arbitrary constant of modulus 1 changes θ_1 and θ_2 with the same constant, so that in the remainder to obtain φ this constant disappears. In other words, the vector u is uniquely defined, and the map is now indeed one-to-one.

4. THE ACTION OF THE SPIN-1/2 OPERATORS ON THE POINCARÉ SPHERE

4.1. Physical Assumptions

We have now made clear the connection between an element of the complex space $\mathbb{C}^2$ and an element of the space of reals $\mathbb{R}^3$. There also exists a connection between the operators in the complex space $\mathbb{C}^2$ and the operators on the Poincaré sphere. Our guideline is that the averages of a physical observable are independent of our description (thus independent of whether we are describing it in the complex space $\mathbb{C}^2$ or on the Poincaré sphere) and the connection that exists between the in-product in $\mathbb{C}^2$ and the scalar product of vectors on the Poincaré sphere. More precisely, the average $\overline{S_1(\psi_w)}$ of an observable $S_1 \in \mathbb{C}^4$ when the entity is in a state ψ_w is given by

$$\overline{S_1(\psi_w)} = \langle S_1(\psi_w) \mid \psi_w \rangle$$

and if we denote the action of the spin operator on the Poincaré sphere by

T_1 [thus, with $S_1(\psi_w)$ is associated the vector $T_1(w)$ on the Poincaré sphere], we can write the connection of the inner product which is of interest to us:

$$|\langle S_1(\psi_w)|\psi_w\rangle|^2 = \frac{1 + T_1(w)\cdot w}{2}$$

These two formulas will be sufficient to define and study the action on a state w by a general spin-1/2 measurement e_u^ϵ on the Poincaré sphere. By the word "action" we mean that the square of the average $\overline{T_\epsilon(w)}$ of the operator T_ϵ (corresponding to the general spin-1/2 measurement e_u^ϵ) on the Poincaré sphere when the entity is in a state p_w is given per definition and in analogy with the quantum case by

$$\overline{T_\epsilon(w)}^2 = \frac{1 + T_\epsilon(w)\cdot w}{2}$$

The operator T_1 corresponds to a rotation over π. This is so because the formula shows us that the angle between $T_\epsilon(w)$ and w is twice the angle θ between w and u:

$$\begin{aligned} |\langle S_\epsilon(\psi_w)|\psi_w\rangle|^2 &= \frac{1 + T_\epsilon(w)\cdot w}{2} \\ &= \cos^2\theta \\ &= \frac{1 + \cos 2\theta}{2} \end{aligned}$$

because $\langle S_\epsilon(\psi_w)|\psi_w\rangle$ is the average and this is $\cos\theta$ in the quantum case. Moreover, also the angle between $T_\epsilon(w)$ and u is θ:

$$\begin{aligned} \langle S_1(\psi_w)|\psi_u\rangle &= \langle \psi_w | S_1^\dagger(\psi_u)\rangle \\ &= \langle \psi_w | S_1(\psi_u)\rangle \\ &= \langle \psi_w|\psi_u\rangle \end{aligned}$$

because S_1 is self-adjunct and ψ_u is an eigenvector of S_1 with eigenvalue $+1$. Using elementary triangle inequalities on the sphere, we see thus that S_1 is a rotation over 180 deg. An elementary calculation reveals for an arbitrary

$$u = (\sin\theta_1 \cos\varphi_1, \quad \sin\theta_1 \sin\varphi_1, \quad \cos\theta_1)$$

the following form of the rotation:

$$S_{e^1_{u(\theta_1,\varphi_1)}} = \begin{pmatrix} -1 + 2\cos^2\varphi_1 \sin^2\theta_1 & 2\sin^2\theta_1 \cos\varphi_1 \sin\varphi_1 & 2\cos\varphi_1 \cos\theta_1 \sin\theta_1 \\ 2\sin^2\theta_1 \cos\varphi_1 \sin\varphi_1 & -1 + 2\sin^2\varphi_1 \sin^2\theta_1 & 2\sin\varphi_1 \cos\theta_1 \sin\theta_1 \\ 2\cos\varphi_1 \cos\theta_1 \sin\theta_1 & 2\sin\varphi_1 \cos\theta_1 \sin\theta_1 & \cos 2\theta_1 \end{pmatrix}$$

4.2. Mathematical Assumptions

The physical assumptions only make it possible to define the angle γ_w between a vector w and its image $T_\epsilon(w)$, which will become clearer in a following section. In general this means that we only can say that $T_\epsilon(w)$ lies on a small circle on the Poincaré sphere, centered around the axis $[-w, w]$ and making an angle γ_w with it. Here we can make different choices to define our $T_\epsilon(w)$ unambiguously. We can make the assumption that $T_\epsilon(w)$ makes the same angle θ with u as u with w. This is a completely arbitrary demand. It is inspired by the fact that in the quantum case we have that $(T_1)^2 = id_{\mathbb{C}^2}$. There is always the possibility to "get back where you started from." If we need the possibility to get back from $T_\epsilon(w)$ to w, we have to be aware of the fact that the action of T_ϵ on $T_\epsilon(w)$ depends on the angle $\theta_{T_\epsilon(w)}$ between $T_\epsilon(w)$ and u, and it is necessary that this angle is equal to the angle θ_w between u and w if we want that a same but opposite action is made. If we look at the quantum case more closely, we see that it is in fact the inverse of S_1 which is involved: $S^2 = S \cdot S^{-1} = S^{-1} \cdot S = id_{\mathbb{C}^2}$, because in the quantum case we have that $S^{-1} = S$. Of course in the intermediate cases this property of the spin operator is lost. More precisely, we can say that the operator T_ϵ is split for $\epsilon \neq 1$ into two operators: a "left-handed" T_l (the set $\{T_l(w), w, u\}$ is "left-handed") and a "right-handed" T_r. Then we can restate the formula $(T_1)^2 = id_{surf}$ by $T_r \cdot T_l = T_l \cdot T_r = id_{surf}$. It simply means that for the nonquantum case symmetry is lost.

4.3. Study of the Algebra of Spin-1/2 Measurement Operator T_1 in the Quantum Case

With the general matrix representation

$$S_{e^1_{u(\theta_1,\varphi_1)}}$$

we can imitate consecutive measurements around different arbitrary axes by multiplying their matrix representations and study the consequences. In particular we are interested in whether we have commutativity. We therefore take two arbitrary unit vectors u and w:

$$u = (\sin \theta_1 \cos \varphi_1, \quad \sin \theta_1 \sin \varphi_1, \quad \cos \theta_1)$$
$$w = (\sin \theta_2 \cos \varphi_2, \quad \sin \theta_2 \sin \varphi_2, \quad \cos \theta_2)$$

and their corresponding matrices

$$S_{e^1_{u(\theta_1,\varphi_1)}} \quad \text{and} \quad S_{e^1_{w(\theta_2,\varphi_2)}}$$

given by

$$S_{e^1_{u(\theta_1,\varphi_1)}} = \begin{pmatrix} -1 + 2\cos^2\varphi_1 \sin^2\theta_1 & 2\sin^2\theta_1 \cos\varphi_1 \sin\varphi_1 & 2\cos\varphi_1 \cos\theta_1 \sin\theta_1 \\ 2\sin^2\theta_1 \cos\varphi_1 \sin\varphi_1 & -1 + 2\sin^2\varphi_1 \sin^2\theta_1 & 2\sin\varphi_1 \cos\theta_1 \sin\theta_1 \\ 2\cos\varphi_1 \cos\theta_1 \sin\theta_1 & 2\sin\varphi_1 \cos\theta_1 \sin\theta_1 & \cos 2\theta_1 \end{pmatrix}$$

$$S_{e^1_{w(\theta_2,\varphi_2)}} = \begin{pmatrix} -1 + 2\cos^2\varphi_2 \sin^2\theta_2 & 2\sin^2\theta_2 \cos\varphi_2 \sin\varphi_2 & 2\cos\varphi_2 \cos\theta_2 \sin\theta_2 \\ 2\sin^2\theta_2 \cos\varphi_2 \sin\varphi_2 & -1 + 2\sin^2\varphi_2 \sin^2\theta_2 & 2\sin\varphi_2 \cos\theta_2 \sin\theta_2 \\ 2\cos\varphi_2 \cos\theta_2 \sin\theta_2 & 2\sin\varphi_2 \cos\theta_2 \sin\theta_2 & \cos 2\theta_2 \end{pmatrix}$$

and study the cases where these two matrices commute. It is easy to see that there only will be problems on the nondiagonal elements of the matrix product. Moreover, we only need to calculate three of these nondiagonal elements, because the other three elements will contribute no further conditions on u and w. This is so because

$$(S_{e^1_{u(\theta_1,\varphi_1)}} \cdot S_{e^1_{w(\theta_2,\varphi_2)}})^t = S^t_{e^1_{w(\theta_2,\varphi_2)}} \cdot S^t_{e^1_{u(\theta_1,\varphi_1)}}$$
$$= S_{e^1_{w(\theta_2,\varphi_2)}} \cdot S_{e^1_{u(\theta_1,\varphi_1)}}$$

so that commutativity is obtained if and only if the matrix

$$S_{e^1_{u(\theta_1,\varphi_1)}} \cdot S_{e^1_{w(\theta_2,\varphi_2)}}$$

is equal to its transponent. After some elementary calculations this leads to the following set of conditions on u and w:

$$\begin{cases} (u\cdot w)\sin(\varphi_1 - \varphi_2)\sin\theta_1 \sin\theta_2 = 0 \\ (u\cdot w)(\cos\varphi_1 \sin\theta_1 \cos\theta_2 - \cos\varphi_2 \sin\theta_2 \cos\theta_1) = 0 \\ (u\cdot w)(\sin\varphi_1 \sin\theta_1 \cos\theta_2 - \sin\varphi_2 \sin\theta_2 \cos\theta_1) = 0 \end{cases}$$

so that we see that a sufficient condition for commutativity is that u and w are orthogonal. If they are not, we can divide by their scalar product, which is now nonzero, and we obtain the following relations:

$$\begin{cases} \sin(\varphi_1 - \varphi_2)\sin\theta_1 \sin\theta_2 = 0 \\ \cos\varphi_1 \sin\theta_1 \cos\theta_2 - \cos\varphi_2 \sin\theta_2 \cos\theta_1 = 0 \\ \sin\varphi_1 \sin\theta_1 \cos\theta_2 - \sin\varphi_2 \sin\theta_2 \cos\theta_1 = 0 \end{cases}$$

from which we can deduce that either $u = w$ or else that $u = -w$, thanks to the need for θ_1 and θ_2 to have a value between 0 and π. Only then do we see that the product matrix

$$S_{e^1_{u(\theta_1,\varphi_1)}} \cdot S_{e^1_{w(\theta_2,\varphi_2)}}$$

becomes equal to its transponent, in perfect correspondence with the description in Hilbert space of course.

5. THE GENERALIZED SPIN-1/2 MEASUREMENT OF THE ϵ-MODEL

5.1. Satisfying the Physical Assumptions

We will study the spin operator for various values of ϵ. For $\epsilon = 1$ we have found that the action of the spin operator is that of a rotation over π. Because the state space is in general not a Hilbert space we use the following correspondence:

$$\frac{1 + T_\epsilon(w)\cdot w}{2} = \overline{T_\epsilon(w)}^2$$

and give to the outcome x^1_u the numerical value of $+1$ and to the outcome x^2_u the numerical value of -1. Doing this, we can calculate the average $\overline{T_\epsilon(w)}$ by means of the given probabilities:

$$\overline{T_\epsilon(w)} = (+1)P^\epsilon(p_u \mid p_w) + (-1)P^\epsilon(p_{-u} \mid p_w)$$

and can be found to depend upon the angle θ between w and the spin direction u in the following way:

1. If $\cos\theta \geq \epsilon$, then $\overline{T_\epsilon(w)} = 1$.
2. If $\epsilon \geq \cos\theta \geq -\epsilon$, then $\overline{T_\epsilon(w)} = (\cos\theta)/\epsilon$.
3. If $\cos\theta \leq -\epsilon$, then $\overline{T_\epsilon(w)} = -1$.

So we find that:

1. If $\cos\theta \geq \epsilon$, then $[1 + T_\epsilon(w)\cdot w]/2 = 1$.
2. If $\epsilon \geq \cos\theta \geq -\epsilon$, then $[1 + T_\epsilon(w)\cdot w]/2 = (\cos^2\theta)/\epsilon^2$.
3. If $\cos\theta \leq -\epsilon$, then $[1 + T_\epsilon(w)\cdot w]/2 = 1$.

If we now make the assumption that $T_\epsilon(w)$ is of unit length, we can calculate the angle γ_w between $T_\epsilon(w)$ and w:

1. If $\cos\theta \geq \epsilon$ or $\cos\theta \leq -\epsilon$, then $\cos\gamma_w = 1$.
2. If $\epsilon \geq \cos\theta \geq -\epsilon$, then $\cos\gamma_w = (2\cos^2\theta - \epsilon^2)/\epsilon^2$.

It is obvious that this indeed reduces to a rotation over 180 deg if $\epsilon = 1$. This is so because the formula shows that the angle between $T_1(w)$ and w is twice the angle θ between w and u. Moreover, the angle between $T_1(w)$ and u is also θ, as we proved earlier for the quantum case and demanded for the intermediate cases.

For $\epsilon = 1$ we see that the operator T_ϵ is linear:

$$T_1\left(\frac{w + v}{\sqrt{2}}\right) = \frac{T_1(w) + T_1(v)}{\sqrt{2}}$$

This is trivial if we write the action of T_1 down in some more geometrical way, making clear its linearity: $T_1(w) = -w + 2(u\cdot w)u$. This result makes it a natural question to ask under which circumstances the operator T_ϵ is linear.

5.2. The Intermediate Cases ($0 \neq \epsilon \neq 1$)

Under which circumstances is this T_ϵ a linear operator on the Poincaré sphere? We will show that there is no such operator.

Theorem 1. T_ϵ is a linear operator if and only if $\epsilon = 1$.

Proof. We have to show that $T_\epsilon(w + v) = T_\epsilon(w) + T_\epsilon(v)$ for every v, w on the sphere. Because we supposed that T_ϵ is a linear operator on the Poincaré sphere, we will check that

$$\frac{T_\epsilon(w) + T_\epsilon(v)}{\sqrt{2}} = T_\epsilon\left(\frac{w + v}{\sqrt{2}}\right)$$

and this is sufficient if T_ϵ is linear.

Let us take $w = u$ and $v = u_\perp$, an arbitrary vector orthogonal to u. It is obvious that $\cos\gamma_w = 1$ because θ_{uw} (the angle between u and w) $= 0$ and $\cos\gamma_v = -1$ because $\theta_{uv} = \pi/2$. In other words, a vector v on the equator of u will always be mapped onto its antipode $-v$, and u will always be mapped by T_ϵ onto itself. More precisely, if we define the eigensets

$eig_u^\epsilon(1)$ and $eig_u^\epsilon(2)$ as the sets $eig_u^\epsilon(1) = \{p_v | \epsilon \leq v \cdot u\}$ and $eig_u^\epsilon(2) = \{p_v | v \cdot u \leq -\epsilon\}$, then every vector of an eigenset will be unchanged by the operator. We find that

$$\frac{T_\epsilon(w) + T_\epsilon(v)}{\sqrt{2}} = \frac{w - u_\perp}{\sqrt{2}}$$

We have now two possibilities for $(w + v)/\sqrt{2}$:

1. If $(w + v)/\sqrt{2}$ is an eigenset, it will be mapped onto itself, and T_ϵ is clearly not linear.

2. $(w + v)/\sqrt{2}$ is not in an eigenset, and the action of T_ϵ will transform it into $T_\epsilon((w + v)/\sqrt{2})$. The angle θ between $(w + v)/\sqrt{2}$ and u is $\pi/4$. So we find

$$\cos \gamma_{(w+v)/\sqrt{2}} = \frac{2\cos^2\theta - \epsilon^2}{\epsilon^2} = \frac{1 - \epsilon^2}{\epsilon^2}$$

On the other hand, we see that the angle between $(w - u_\perp)/\sqrt{2}$ and $(w + v)/\sqrt{2}$ is $\pi/2$ (they are orthogonal) such that we also have that $\cos \gamma_{(w+v)/\sqrt{2}} = 0$. This can only be the case if and only if $\epsilon = 1$. Thus only in the quantum case do we find that T_ϵ is linear.

5.3. The Classical Case

The classical case needs special study, because now $\epsilon = 0$, making it impossible to follow the guidelines of the previous sections. Let us recall that if $\kappa = v \cdot u$, the experiment e_u^ϵ gives outcome x_u^1 with probability 1/2 and outcome x_u^2 with probability 1/2. Thus the average $\overline{T_0(w)}$ for an arbitrary w on the equator $(w \cdot u = 0)$ is

$$\overline{T_0(w)} = (+1)P^0(p_u | p_w) + (-1)P^0(p_{-u} | p_w) = (+1)\tfrac{1}{2} + (-1)\tfrac{1}{2} = 0$$

Hence $[1 + T_0(w) \cdot w]/2 = 0$, and the angle γ_w between w and $T_0(w)$ is π.

We have two possibilities:

1. We neglect the foregoing remark and define T_0 as the identity for $\epsilon = 0$, making an extrapolation of the eigensets. Indeed, the upper open half-sphere is an eigenset and the lower open half-sphere is an eigenset of T_0. Making the eigensets closed by defining the action of T_0 on the equator as the identity, we recover a linear operator. Unfortunately, doing this, we not only abandon the guidelines we set out with in the beginning, but we also lose symmetry and the eigensets get mixed up, which is an untenable situation.

2. That T_0 is the identity on the whole sphere, except on the equator, where it maps the points on their antipodes. We see that we have no linearity

in the classical case, too. It seems the most natural thing to do, maintaining our guidelines and respecting symmetry.

6. THE ASYMMETRIC AND GENERAL ASYMMETRIC CASE

The reasoning above can be repeated for an even more general model: the asymmetric ϵ-model in which an extra parameter d is introduced, $d \in [-1 + \epsilon, 1 - \epsilon]$, describing this asymmetry. If d equals zero, we find the previous case. For $d \neq 0$, no linearity will be found, as we shall prove below. We will use the notation T_ϵ^d to denote the corresponding operator.

6.1. The ϵ,d-Model

The probability $P_d^\epsilon(p_u | p_w)$ to obtain an outcome $x_u^1 = +1$ is given by:

1. If $u \cdot w \geq d + \epsilon$, $P_d^\epsilon(p_u | p_w) = 1$.
2. If $d + \epsilon \geq u \cdot w \geq d - \epsilon$, $P_d^\epsilon(p_u | p_w) = \frac{1}{2\epsilon}(w \cdot u - d + \epsilon)$.
3. If $d - \epsilon \geq u \cdot w$, $P_d^\epsilon(p_u | p_w) = 0$.

The probability $P_d^\epsilon(p_{-u} | p_w)$ to obtain an outcome $x_u^2 = -1$ is given by:

1. If $u \cdot w \geq d + \epsilon$, $P_d^\epsilon(p_{-u} | p_w) = 0$.
2. If $d + \epsilon \geq u \cdot w \geq d - \epsilon$, $P_d^\epsilon(p_{-u} | p_w) = \frac{1}{2\epsilon}(d + \epsilon - w \cdot u)$.
3. If $d - \epsilon \geq u \cdot w$, $P_d^\epsilon(p_{-u} | p_w) = 1$.

Thus we have after a completely analogous calculation to the previous sections that the angle γ_w between $T_\epsilon^d(w)$ and w is given by:

1. If $u \cdot w \geq d + \epsilon$, $\cos \gamma_w = 1$.
2. If $d + \epsilon \geq u \cdot w \geq d - \epsilon$, $\cos \gamma_w = [2(u \cdot w - d)^2 - \epsilon^2]/\epsilon^2$.
3. If $d - \epsilon \geq u \cdot w$, $\cos \gamma_w = 1$.

6.2. Satisfaction of the Mathematical Assumption

Can now the left spin operator T_ϵ^d, which we denote by $T_{\epsilon,l}^d(w)$, be chosen in a way such that $T_{\epsilon,l}^d(T_{\epsilon,r}^d(w))$ is again w? This is not always the case. For an arbitrary w the calculations show that $T_{\epsilon,l}^d(w)$ lies on a circle which makes an angle γ with w, which depends on the angle θ_w between w and u. So, to go back to w we need that $T_{\epsilon,r}^d(w)$ makes the same angle θ_w with u. But for w with $u \cdot w = d$, we see that $\cos \gamma = -1$, so that $T_{\epsilon,r}^d(w) = -w$ and

$$\theta_{T_{\epsilon,r}^d(w)} = \pi - \theta_w$$

such that $T_{\epsilon,l}^d(T_{\epsilon,r}^d(w)) \neq w$, hence $T_{\epsilon,l}^d \cdot T_{\epsilon,r}^d \neq id_{surf}$. A simple calculation shows that for $d = 0$, it is always possible to choose $T_{\epsilon,l}^d(w)$ such that

$T^d_{\epsilon,l}(T^d_{\epsilon,r}(w)) = w$. One can see this easily by considering the fact that the "troublesome" case w with $u \cdot w = d = 0$ gives no problems anymore because now $T^d_{\epsilon,r}(w) = -w$ lies also on the operator and no longer under it. So even in that extreme case no further problems arise. The reasoning for $T^d_{\epsilon,r} \cdot T^d_{\epsilon,l} \neq id_{surf}$ is completely analogous.

6.3. Linearity for the Asymmetric Situation

It can be shown that an asymmetric situation can never give rise to a linear spin operator. If T^d_ϵ is a linear function which maps the Poincaré sphere onto itself, d must equal zero.

Theorem 2. T^d_ϵ is a linear function which maps the Poincaré sphere onto itself $\Rightarrow d = 0$.

Proof. If T^d_ϵ is linear, we must have $\forall v, w$: $T^d_\epsilon(v) + T^d_\epsilon(w) = T^d_\epsilon(v + w)$ and $T^d_\epsilon(\alpha \cdot v) = \alpha \cdot T^d_\epsilon(v)$, $\forall \alpha \in \mathbb{R}$. In particular, for $v = u$, and w such that $\cos\theta_w = d$, we have that $T^d_\epsilon(v) = u$ and $T^d_\epsilon(w) = -w$. The norm of

$$v + w = \sqrt{2(1 + \cos\theta_w)} = \sqrt{2(1 + d)}$$

So $(v + w)/\sqrt{2(1 + d)}$ has norm 1 and lies on the sphere. Its image also lies on the sphere by assumption and thus has norm one.

But

$$T^d_\epsilon\left(\frac{v + w}{\sqrt{2(1 + d)}}\right) = \frac{T^d_\epsilon(v) + T^d_\epsilon(w)}{\sqrt{2(1 + d)}} = \frac{v - w}{\sqrt{2(1 + d)}}$$

which has norm $\sqrt{2(1 - d)}/\sqrt{2(1 + d)}$. This can only be 1 for $d = 0$.

7. CONCLUSION

We have constructed a set of operators of the generalized ϵ,d-model and investigated the circumstances under which these operators become linear. It was shown that only for the symmetric case with maximal fluctuations of the interaction between entity and measurement apparatus do the operators become linear on the Poincaré sphere. This case coincides with ordinary quantum mechanics, where the set is the generating set of a *-algebra. In a later article we will investigate further the structure of the set of general operators of the ϵ,d-model.

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